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Automorphisms and Weighted Values[1]

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Abstract: The notion of automorphism is an essential tool to capture the meaning of any mathematical structure. We apply this idea to cooperative games and obtain two interesting characterizations of the automorphisms of such a game: the one, in the complete case, as the permutations of players which preserve the (classical) Shapley value; the other, for the general case, as the permutations preserving *all* weighted Shapley values.

1 Rationale

In game theory, the following question seems of interest: given a game (in characteristic function form), under what circumstances can we say that the game treats all players (or, perhaps, some particular pair of players) equally? We might imagine some tournament director, trying to organize the way a game is scheduled, so that, in the end, no losing player might be able to claim that he lost "because of the way the tournament was organized".

Let us say a game is *fair* if it treats all its players equally. Then the following two criteria seem reasonable:

(a) a *sufficient* condition for game v to be fair is that $v(S)$ depend only on the cardinality of S.

(b) A *necessary* condition for game v to be fair is that some accepted measure, say the Shapley value, assign equal payoffs to all players.

Obviously, the first condition (symmetry) implies the second (equality of the Shapley value), but the converse is not true. It is the intermediate cases, i.e. games v such that $\Phi_i[v] = \Phi_j[v]$ for all pairs, but which are not symmetric, that seem most interesting.

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The following two examples, from "real" games, might make the situation clear.

Consider, first, the game of bridge. In such a game, the cards dealt will certainly make a difference; the position (i.e. whether one deals or not) possibly also makes a difference. Now in a tournament, the difference in cards (the luck of the deal) can be more or less obviated by duplication of hands; the difference in position is obviated by rotating the dealership. With this in mind, much of the influence of "luck" has been removed. In theory, then, a bridge tournament would seem to be "fair" in every sense of the word.

Suppose, however, that someone is competing in a *singles* bridge championship. This means an individual, without a partner, is up against all others. In a large field, it may be impossible to consider all pairings, simply because of time constraints. Thus a losing player might feel (and might indeed be correct in feeling) that she lost, not because of inferior play, but because the luck of the draw gave her an above-average number of weak partners. (In an analogous extreme case, we might think of a player who refuses to play at a certain club because most players already have their set partners, and she finds herself always having the same, very weak, partner.)

In the second place, consider a tennis tournament, played (as most are) according to the single elimination rule. One hundred twenty-eight players play seven rounds, in each of which half the remaining players lose, so that the seventh round consists of only one game between two players.

In the abstract, once again, this process seems perfectly fair. The strongest player will, almost by definition, win each of his seven matches and thus win the championship. This does not, however, take into account the fact that, in each match, the stronger player is far from certain to win: stronger usually means, merely, that he has a better than 1/2 probability of winning the match.

To see the effects of this, suppose there are three strong players, called G, S, and M respectively, and many (125) weak players. Suppose that, in any pairing between G, S, and M, either one of the pair has probability 1/2 of winning, and, in any pairing between one of these three and one of the remaining players, G, S, or M has probability close to 1 of winning.

The tournament director must now decide on the schedule. Typically, G, S, and M will be seeded 1, 2, and 3, in some order. Assume they are in fact seeded in that order. Then, in all probability, S and M will meet in one of the semi-finals; the winner between these two will meet G in the finals. It is not difficult to see that, in such case, G has probability 1/2 of winning, while S and M each have probability 1/4 of winning. (This does not even take account of the possibility that in the semi-finals, S and M might wear each other out, leaving the victor to play a relatively fresh G.) Thus, the draw definitely favours G, no matter how fair the tournament might, in the abstract, seem.

In cooperative game theory, a similar situation might well hold. In game v, assume two players, i and j, have equal chances, in the sense that $\Phi_i[v] = \Phi_j[v]$. Nevertheless, it may be that, to realize fully his potential, i needs to make a coalition with an unpredictable player, h; j is not faced with such a problem. If

h does, indeed, act in strange fashion, player i might find herself doing poorly, and might well be justified in complaining that j did better (than i) because i needed to deal with a player (h) that acted irrationally.

In a purely symmetric game (i.e. $v(S)$ depends only on the cardinality of S), i cannot logically make this complaint, as, anything j did, i could just as easily have done. In other games – even games with transitive automorphism groups – this might well happen. We prove, in this article, that – assuming this "strange" behavior can be modelled by differential weights in the weighted Shapley value – the only truly "fair" games are the symmetric games.

2 Automorphisms and Desirability in Games

The notion of automorphism is an essential tool in almost all branches of Mathematics concerned with some kind of structures. In the case of cooperative games, automorphisms are those one-to-one transformations of the set of players which leave invariant the characteristic function and, hence, do not alter the "in abstracto" strategic situation.

A natural way to compare the strategic positions of any two players is the use of the desirability and indifference relations introduced by Isbell (1956). The obvious advantages found in the case that desirability is total (complete) give us some justification for calling the games where it occurs, complete games.

On the other hand, the Shapley value is a numerical measure – derived from an axiomatic procedure (Shapley, 1953) – which evaluates the differences between the players' strategic capabilities. By generalizing it to the so-called weighted Shapley values (see, e.g. Kalai and Samet, 1987), one is able to add to the game exogenous considerations that very often influence the players' behavior.

An earlier paper (Carreras, 1984), studied the group of automorphisms of any complete simple game. The results there are easily extended to the class of complete cooperative games. Sections 2 and 3 are devoted to summarizing these, showing interesting relationships between the group of automorphisms, the desirability and indifference relations, and the (classical) Shapley value. The main result (Theorem 3.3 in this paper) characterizes the automorphisms of such a game as the permutations of players which preserve the Shapley value, that becomes, then, the *characteristic invariant* for automorphisms.

Completeness is a sufficient condition to obtain this result, but it is not a necessary one: any game "irregular" enough to guarantee that all players have distinct values trivially satisfies the basic relationship between automorphisms and the value. Hence, it does not seem appropriate to ask for a necessary and sufficient condition by relaxing completeness. In Section 4 we directly attack the general case of cooperative games and, using a recent theorem on separation of games by weighted values (Carreras and Owen, 1993), we characterize (in

Theorem 4.3) the automorphisms of any cooperative game as those permutations of players which preserve *all* the weighted Shapley values.

Let $N=\{1,2,\ldots,n\}$ be a fixed set of players, and let $\mathscr{G}_N$ be the linear space of all cooperative games on N, i.e. functions $v:2^N \to \mathbb{R}$ such that $v(\phi)=0$.

Let S_n be the symmetric group on N. Every permutation $\theta \in S_n$ operates on games through the map $\theta^*:\mathscr{G}_N \to \mathscr{G}_N$ defined by

$$(\theta^* v)(S)=v(\theta^{-1}S) \quad \forall S \subseteq N, \quad \forall v \in \mathscr{G}_N.$$

This is equivalent to say that $(\theta^* v)(\theta S)=v(S)$ for any $S \subseteq N$; thus, the role played by θi (respectively θS) in game $\theta^* v$ is identical to the role played by i (resp. S) in v.

θ^* is a linear and bijective map, and $(\theta^*)^{-1}=(\theta^{-1})^*$. Moreover, $(\theta\psi)^*=\theta^*\psi^*$ if $\psi \in S_n$, and $\theta^*=\mathrm{id}$ (on $\mathscr{G}_N$) if, and only if, $\theta=\mathrm{id}$ (on N). In other words, the map $\theta \mapsto \theta^*$ gives an injective linear representation $S_n \to \mathrm{Aut}(\mathscr{G}_N)$.

Let $v \in \mathscr{G}_N$. Its *orbit* $O(v)=\{w=\theta^* v$ for some $\theta \in S_n\}$ is the set of games on N which are *isomorphic* to v (identical to v up to rearrangements of positions). Its *isotropy group* is

$$\mathrm{Aut}(v)=\{\theta \in S_n/\theta^* v=v\},$$

and every $\theta \in \mathrm{Aut}(v)$ will be called an *automorphism* of v. This is equivalent to saying, simply, that $v(\theta S)=v(S)$ for every $S \subseteq N$.

We will use in Section 4 *weight vectors* $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{R}^n$. Every $\theta \in S_n$ operates also on vectors, through the map $\theta^*:\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(\theta^*\alpha)_i=\alpha_{\theta^{-1}i} \quad \forall i \in N, \quad \forall \alpha \in \mathbb{R}^n.$$

This is equivalent to saying that $(\theta^*\alpha)_{\theta i}=\alpha_i$ for all $i \in N$ and $\alpha \in \mathbb{R}^n$, and so the weight of player θi in vector $\theta^*\alpha$ coincides with that of player i in α.

We will abuse notation and write θv (respectively $\theta\alpha$) instead of $\theta^* v$ (resp. $\theta^*\alpha$).

Theorem 2.1: Let $v \in \mathscr{G}_N$. Then:

(a) $\mathrm{Aut}(v)=S_n$ if, and only if, v is a symmetric game (i.e. $v(S)=v(T)$ when $|S|=|T|$).

(b) If v is isomorphic to w, their groups of automorphisms are conjugate:

$$\mathrm{Aut}(v)=\theta^{-1}\,\mathrm{Aut}(w)\theta \quad \text{if } w=\theta v.$$

(c) $[S_n:\mathrm{Aut}(v)]$, the index of $\mathrm{Aut}(v)$ as a subgroup of S_n, coincides with $|O(v)|$ and gives then the number of games on N which are isomorphic to v.

Proof: This proof is straightforward. See also Carreras (1984). □

Remarks 2.2:

(1) As Theorem 2.1(a) shows, the group does not, in general, determine the game.

(2) The converse of 2.1(b) is not true: let $v = u_N$, the unanimity game on N, and let $w = v^*$, the dual of v. (The *dual* v^* of a game v is defined by $v^*(S) = v(N) - v(N - S)$ for every $S \subseteq N$.) Since both games are symmetric, their groups of automorphisms coincide with S_n, which is self-conjugate under any permutation θ; nevertheless, those games are not isomorphic (we also notice that $\mathrm{Aut}(v^*) = \mathrm{Aut}(v)$ for any game v).

Denote by $\Phi: \mathscr{G}_N \to \mathbb{R}^n$ the Shapley value. The original Shapley symmetry axiom can be stated as follows:

$$\Phi_{\theta i}[\theta v] = \Phi_i[v], \quad \forall i \in N, \quad \forall v \in \mathscr{G}_N, \quad \forall \theta \in S_n.$$

As an immediate consequence we obtain

Proposition 2.3: For any game v, the Shapley value $\Phi[v]$ is invariant under automorphisms, that is

$$\Phi_{\theta i}[v] = \Phi_i[v], \quad \forall i \in N, \quad \forall \theta \in \mathrm{Aut}(v). \quad \square$$

3 Complete Games: The (Classical) Shapley Value

Given $v \in \mathscr{G}_N$, a binary relation D is defined on N as follows:

$$iDj \quad \text{(def)} \quad v(S \cup \{i\}) \geq v(S \cup \{j\}) \quad \forall S \subseteq N - \{i,j\}$$

(iDj is to read: i is *more-* or, at least, *not less-desirable than* j as a coalition partner). It is easy to see that D is reflexive and transitive – a preorder. Thus, the lack of antisymmetry is solved by introducing the associated equivalence relation I, defined by

$$iIj \quad \text{(def)} \quad iDj, \quad jDi,$$

so that iIj means that $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq N - \{i,j\}$ (i and j are *indifferent* as partners). As is well known, D induces an ordering in the quotient set, formed by the I-classes. It is easy to see that iIj is equivalent to saying that the transposition t_{ij} is an automorphism of v.

The other basic problem with desirability is that it is not always total. If any two players are comparable by D, we will say that v is a *complete* game: in this case, the I-classes are linearly ordered.

Example 3.1: A game v is a *weighted majority game* if there exist a vector $w \in \mathbb{R}^n$ and a number q such that

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

All weighted majority games are complete, since $w_i \geq w_j$ implies iDj; thus, $w_i = w_j$ implies iIj.

For any game $v \in \mathcal{G}_N$, its group $\mathrm{Aut}(v)$, as a subgroup of S_n, operates on N. Define the *orbit* of any player i as $O_i(v) = \{j = \theta i \text{ for some } \theta \in \mathrm{Aut}(v)\}$. The (distinct) v-orbits give a partition of N, and every orbit is a union of I-classes which have the same cardinality. By Proposition 2.3, the Shapley value $\Phi_i[v]$ is, as a function of i, constant on every orbit; in particular, iIj implies that $\Phi_i[v] = \Phi_j[v]$ (see also Lemma 3.2). The basic difference between a pair of players i, j belonging to the same I-class and a pair of players lying within the same orbit is that, in the first case, the pure transposition t_{ij} evidences that they are in equivalent positions, whereas in the second case a more complex automorphism θ is needed to see this, probably moving simultaneously some of (or all) the remaining players (see Example 4.5).

If there is only one orbit, the group – and the game itself – is said to be *transitive*. In this case the Shapley value is constant.

We state, next, our main results on complete games (Theorem 3.3 and Corollary 3.4). As for Lemma 3.2, which preceeds them and applies to any game, proofs and essentially analogous to those found in Carreras (1984) for the simple case, and hence they will be omitted.

Lemma 3.2: Let $v \in \mathcal{G}_N$ and $i, j \in N$ be such that iDj. Then, $\Phi_i[v] \geq \Phi_j[v]$. If, moreover, $j\not{D}i$, then $\Phi_i[v] > \Phi_j[v]$. $\square$

Theorem 3.3: The automorphisms of a complete game $v \in \mathcal{G}_N$ are the permutations of players which leave invariant the Shapley value of the game, i.e.

$$\mathrm{Aut}(v) = \{\theta \in S_n / \Phi_{\theta i}[v] = \Phi_i[v] \quad \forall i \in N\}.$$

Moreover, if $C_1, C_2, \ldots, C_k$ are the I-classes of v,

$$\mathrm{Aut}(v) = S_{c_1} \times S_{c_2} \times \ldots \times S_{k_k}. \quad \square$$

Corollary 3.4: Let $v \in \mathcal{G}_N$ be a complete game. Then:

(a) v-orbits and I-classes coincide.
(b) $\Phi_i[v] = \Phi_j[v]$ if, and only if, iIj holds.
(c) The Shapley value takes distinct values over distinct v-orbits.
(d) The combinatorial number $\binom{n}{c_1, c_2, \ldots, c_k}$ gives the number of games on N isomorphic to v.

(e) If $\Phi_i[v]$ is constant as a function of i (in particular: if v is transitive) then v is a symmetric game, and whence $\mathrm{Aut}(v) = S_n$, which becomes the only transitive group of a complete game. $\square$

Remark 3.5: Completeness is a reasonable sufficient condition for the Shapley value to be the *characteristic invariant* for automorphisms. Example 4.5 shows that this cannot be guaranteed when the game is not complete. At the same time, the reader may provide some game irregular enough so that all players have distinct values: the group of automorphisms reduces, then, to $\{\mathrm{id}\}$, showing that completeness is far from being necessary.

Given $v \in \mathscr{G}_N$, let

$$O_i = O_i(v) = C_i^1 \cup C_i^2 \cup \cdots \cup C_i^{k_i}, \quad i = 1, 2, \ldots, r$$

be the distinct v-orbits, expressed as unions of I-classes. Let $G(\Phi[v]) = \{\theta \in S_n / \Phi_{\theta i}[v] = \Phi_i[v] \quad \forall i \in N\}$ be the group of permutations of N that preserve $\Phi[v]$, and denote by $S(X)$ the group of permutations of any v-orbit or I-class X. Then, we have

$$\prod_{i,j} S(C_i^j) \subseteq \mathrm{Aut}(v) \subseteq \prod_{i=1}^{r} S(O_i) \subseteq G(\Phi[v]).$$

Completeness implies that these four groups coincide; nevertheless, it does not seem an easy task to find a *weaker* (and interesting) condition equivalent to

$$\mathrm{Aut}(v) = G(\Phi[v]),$$

the equality obtained in Theorem 3.3.

4 The General Case: Weighted Values

We consider again the problem of characterizing the automorphisms of a game, now dropping completeness.

An equivalent form of Theorem 3.3 is as follows: if $v \in \mathscr{G}_N$ is a complete game

$$\theta \in \mathrm{Aut}(v) \Leftrightarrow \Phi_{\theta i}[v] = \Phi_i[v] \quad \forall i \in N.$$

Our main result in this section (Theorem 4.3) will take a similar form, using weighted values instead of the classical value.

Lemma 4.1: The weighted Shapley value $\Phi^\alpha : \mathscr{G}_N \to \mathbb{R}^n$ associated to any $\alpha \in \mathbb{R}^n$ satisfies a "generalized symmetry axiom":

$$\Phi_{\theta i}^{\theta \alpha}[\theta v] = \Phi_i^\alpha[v], \quad \forall i \in N, \quad \forall v \in \mathscr{G}_N, \quad \forall \theta \in S_n.$$

Proof: If we fix i, α and θ, the expression

$$\delta(v) = \Phi_{\theta i}^{\theta\alpha}[\theta v] - \Phi_i^{\alpha}[v]$$

depends linearly on v; thus, it suffices to check the equality for $v = u_S$, the unanimity game with any $S \subseteq N$ as a carrier. From $\theta u_S = u_{\theta S}$ and applying the standard computation of weighted values for unanimity games (Kalai and Samet, 1987) it follows that

$$\Phi_{\theta i}^{\theta\alpha}[\theta u_S] = \begin{cases} \dfrac{(\theta\alpha)_{\theta i}}{\sum_{\theta j \in \theta S} (\theta\alpha)_{\theta j}} & \text{if} \quad \theta i \in \theta S \\ 0 & \text{if} \quad \theta i \notin \theta S. \end{cases}$$

Recalling the way θ operates on games, coalitions and weight vectors, this reduces to

$$\Phi_{\theta i}^{\theta\alpha}[\theta u_S] = \begin{cases} \dfrac{\alpha_i}{\sum_{j \in S} \alpha_j} & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S. \end{cases}$$

and thus $\Phi_{\theta i}^{\theta\alpha}[\theta u_S] = \Phi_i^{\alpha}[u_S]$. □

The following lemma is from Carreras and Owen (1993).

Lemma 4.2: Given any two distinct games v and v' over the set N, there is some weight vector α such that

$$\Phi^{\alpha}[v] \neq \Phi^{\alpha}[v'].$$

Proof: Let $w = v - v' \neq 0$. Let $K \subseteq N$ be such that $w(K) \neq 0$ and, for all $T \subset K$, $w(T) = 0$. We write

$$w = \sum_{\Phi \neq S \subseteq N} c_S(w) u_S$$

where u_S is the unanimity game with carrier S, and

$$c_S(w) = \sum_{T \subseteq S} (-1)^{s-t} w(T).$$

Then $c_K(w) = w(K) \neq 0$, and $c_T(w) = 0$ for all $T \subset K$.

Choose now $\alpha_j = 1$ for all $j \in K$ and α_j very large for $j \in N - K$. Then, for any $i \in K$,

$$\Phi_i^{\alpha}[w] = \frac{c_K(w)}{k} + \sum_{S \neq K} \frac{c_S(w)}{\sum_{j \in S} \alpha_j}.$$

In this last sum, $c_T(w) = 0$ for all $T \subset K$, and, for all other S, the denominator $\sum_{j \in S} \alpha_j$ is very large as S contains at least one $j \in N - K$. Thus the summands can be made arbitrarily small, i.e. $\Phi_i^\alpha[w]$ is the sum of the non-zero term $c_K(w)/k$ and finitely many arbitrarily small terms. It follows that $\Phi_i^\alpha[w]$ must have the same sign as $c_K(w)$. We therefore have

$$\Phi_i^\alpha[v - v'] \neq 0$$

and by linearity,

$$\Phi_i^\alpha[v] \neq \Phi_i^\alpha[v']$$

as desired. $\square$

Theorem 4.3: Let $v \in \mathcal{G}_N$. Then

$$\theta \in \mathrm{Aut}(v) \Leftrightarrow \Phi_{\theta i}^{\theta \alpha}[v] = \Phi_i^\alpha[v] \quad \forall i \in N, \quad \forall \alpha \in \mathbb{R}^n.$$

Proof: ($\Rightarrow$) Use $\theta v = v$ and the formula in Lemma 4.1.
($\Leftarrow$) Assume $\theta \notin \mathrm{Aut}(v)$. Then $\theta v \neq v$ and so $v \neq \theta^{-1} v$. By applying the property of separation of games by weighted values (Lemma 4.2) some $\alpha \in \mathbb{R}^n$ exists such that

$$\Phi^\alpha[v] \neq \Phi^\alpha[\theta^{-1} v].$$

Hence, there exists some $i \in N$ such that

$$\Phi_i^\alpha[v] \neq \Phi_i^\alpha[\theta^{-1} v].$$

Using again Lemma 4.1 it follows that

$$\Phi_i^\alpha[\theta^{-1} v] = \Phi_{\theta i}^{\theta \alpha}[v]$$

and we conclude that there exist some $\alpha \in R^n$ and some $i \in N$ such that

$$\Phi_{\theta i}^{\theta \alpha}[v] \neq \Phi_i^\alpha[v]. \quad \square$$

Remark 4.4: The game-separation property may be strengthened (see Carreras and Owen, 1993, Note 2) in the sense that α can be taken arbitrarily close to vector $(1, 1, \ldots, 1)$, whose associated weighted value is precisely the classical Shapley value. In the present context it may be interpreted as saying that, for θ to be an automorphism of the game, it must preserve any deviation from the classical value, in the sense of Theorem 4.3.

Example 4.5: Let $N = \{1, 2, 3, 4, 5\}$ and let v be the (simple) game generated by the circular triads

$$\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 1\}, \{5, 1, 2\}$$

of vertices of a pentagon, that is:

$$v(S)=\begin{cases}1 & \text{if } S \text{ contains some circular triad}\\ 0 & \text{otherwise.}\end{cases}$$

Here the group of automorphisms is $\mathrm{Aut}(v)=D_{10}$, the dihedral group generated by the rotation $1\mapsto 2\mapsto 3\mapsto 4\mapsto 5\mapsto 1$ and the symmetry defined by $s(i)=6-i$ for every $i\in N$. This group does not contain any transposition, and thus all I-classes are singletons; moreover, only iDi for all $i\in N$ is satisfied by D, so that the game is not complete. But it is transitive, and hence the Shapley value is constant and any permutation – automorphic or not – preserves this value: $G(\Phi[v])=S_5\neq \mathrm{Aut}(v)$.

The difference between $G(\Phi[v])$ and $\mathrm{Aut}(v)$ is made clear by considering weighted values: for any $\theta\in G(\Phi[v])-\mathrm{Aut}(v)$ there exists, according to Theorem 4.3, some $\alpha\in\mathbb{R}^n$ – it may be taken arbitrarily close to $(1,1,\dots,1)$ – such that $\theta\notin G(\Phi^\alpha[v])$, where this group is defined similarly to $G(\Phi[v])$ (recall Remark 3.5).

More explicitly: to leave aside, e.g. $\theta=t_{12}$, take

$$\alpha=(1,\ 1.1,\ 1,\ 1,\ 0.9)$$

(any α of the form $\alpha=(\lambda,\ \mu,\ 1,\ 1,\ 1)$ does not work); then, we have

$$\Phi^\alpha_1[v]=0.200,\quad \Phi^{\theta\alpha}_{\theta 1}[v]=\Phi^{\theta\alpha}_2[v]=0.189,$$

where $\theta\alpha=(1.1,\ 1,\ 1,\ 1,\ 0.9)$.

In fact, Theorem 4.3 states that, for any game $v\in\mathscr{G}_N$,

$$\mathrm{Aut}(v)=\bigcap_{\alpha\in\mathbb{R}^n} G(\Phi^\alpha[v]).$$

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